Geometric Quantization

Christopher Miller
bowties@mit.edu

DRP Mentor: Robert Jones
rjones122@gmail.com

MIT - Department of Physics

You are used to quantizing

Canonical Quantization

$$Q(q) = q \qquad\qquad Q(p) = -i\hbar\partial_q$$

Nice properties:

- $[Q(q), Q(p)] = i\hbar \implies$ Uncertainty
- Standard potentials work nicely Ex: $H = p^2/2m + V(q)$

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Bad properties:

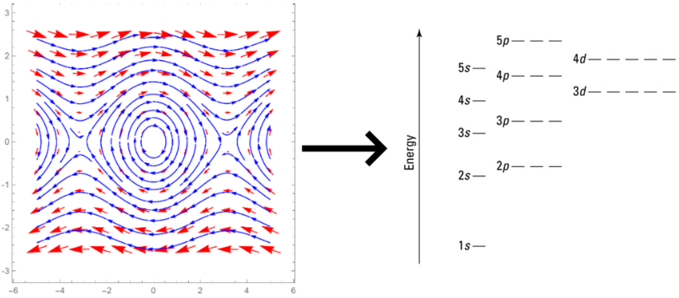
- $Q(qp)$ ambiguous
- Meanwhile, classical symmetries in $C^\infty(M, \omega)$ w/ Poisson

Quantizations should be nice

Definition

A *quantization* is a map Q from smooth functions on a manifold to operators on a Hilbert Space obeying Dirac's axioms:

- $Q(1)$=Identity
- $[Q(A), Q(B)] = i\hbar Q(\{A, B\})$
- Q of complete set of functions is a CSCO



Quantizations are Hard

Theorem (Groenewold-van Hove)

There is no complete quantization of $C^\infty(\mathbb{R}^{2n})$.
Breaks beyond cubic polynomials.
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Solution is to **throw something away**.

Deformation: let $[Q(A), Q(B)] = i\hbar Q(\{A, B\}) + O(\hbar^2)$ - fine for chemists but Ehrenfest would be unhappy.

Geometric: make oversized Hilbert space then throw away pieces (like restricting to $\{f(q)p + g(q)\}$)

More Naive Quantizations

The "I just learned symplectic geometry" quantization

Know $\{A, B\} = \omega(v_A, v_B)$.
Try $Q(f) = v_f$.
Then $Q(\{A, B\}) = v_{\{A,B\}} = [v_A, v_B] = [Q(A), Q(B)]$
But $Q(1) = 0$, no uncertainty.

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The "uncertain" quantization

Try $Q(f) = v_f + f\cdot$
But $[Q(q), Q(p)] = 2 \neq 1 = Q(1)$

Geometric Extensions: Forms and Curvature

The Hove-Segal quantization

Know, locally, $\omega = d\theta$
Try $Q(f) = v_f + f \cdot + \theta(v_f)\cdot$
Works on Hilbert Space $\mathbb{R}^{2n} \to \mathbb{C}$, $(f, g) = \int_M f^* g \omega^n$
But $\partial_q, Q(q), Q(p)$ commute...

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The Kirillov-Konstant-Soriau quantization on (M, ω)

Take prequantum line bundle $U(1) \to Q \xrightarrow{\pi} M$

$$Q(f) = -i\hbar \nabla_{v_f} + f$$

With $\nabla_{v_f} = (d + A)(v_f)$ and curvature $F = dA$,
if $F = \omega$, then

$$[-i\nabla_A + A, -i\nabla_B + B] = ... = -i(-i\nabla_{\{A,B\}} + \{A, B\})$$

Quantumness and Angular Momentum

Note that F is $U(1)$ curvature iff $\int_\alpha F \in 2\pi\mathbb{Z}, \forall\alpha \in H_2(M)$.

Consider space $U(1) \longrightarrow S^3 \xrightarrow{<\vec{\sigma}>} S^2$

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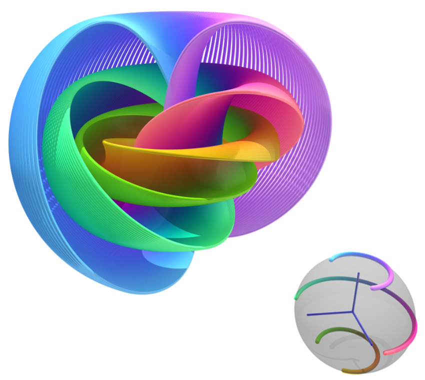

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Consider space $U(1) \longrightarrow S^3 \xrightarrow{<\vec{\sigma}>} S^2$ with Berry Connection

$$A = -is(\langle \partial_\theta \rangle \, d\theta + \langle \partial_\phi \rangle \, d\phi) \qquad \omega = \frac{s}{2}\sin(\theta) d\theta \wedge d\phi \qquad \int_{S^2} \omega = 2\pi s$$

Thus spin quantized.
Recover angular momentum algebra from $-s/2\{x, y, z\}$

Hilbert space: $\frac{1}{f(z)}(z)^{2s}$ is holomorphic $\implies f \in \mathrm{span}\{1, z, ..., z^{2s}\}$,
$z = \tan(\theta/2)e^{i\phi}$

Thank you.
Any Questions?

Messy Ideas

The nonexample:
$Q(\{q^3, p^2\} + \frac{1}{12}\{\{p^2, q^3\}, \{q^2, p^3\}\}) \neq [q^3, p^2] + \frac{1}{12}[[p^2, q^3], [q^2, p^3]]$

Angular Momenta:
$Q(x) = \frac{-s}{2}\cos\phi\tan(\theta/2) + i\hbar\cos\phi\cot\theta\partial_\phi - i\hbar\sin\phi\partial_\theta$
$Q(y) = \frac{-s}{2}\sin\phi\tan(\theta/2) + i\hbar\sin\phi\cot\theta\partial_\phi - i\hbar\cos\phi\partial_\theta$
$Q(z) = -i\hbar\partial_\phi - \frac{s}{2}$

Completeness: Exists a dense set of separately analytic wavefunctions for quantization of irreducible base of Poisson algebra($\sum|Q(b)^k\psi|t^k/k!$ converges in $\mathcal{H}$ for some t for each b).

Can also quantize states directly

Use WKB approximation $\psi \sim A(q)e^{iS(q)/\hbar}$
Evolution via transport requires half-density $A(q)$
Projectible on Lagrangian immersion $\omega_L = 0$ with asymptotic inverse Fourier transform.
Polarizations have issues on overlaps so need Maslov correction.

$$\psi(Q,t) = \sum_{\{q_j : g^t q_j = Q\}} \varphi(q_j) \left| \frac{D[q(g^t(p(q_j), q_j)]}{Dq_j} \right|^{-1/2}$$

$$\exp\left\{ i/\hbar \left[s(q_j) + \int_0^t L(g^t(p(q_i), q_i))dt \right] - \frac{i\pi\mu_j}{2} \right\}$$

Where $S(q) = (Sq, q)/2$ and μ_j sign flips in $(0, t]$ of $det(I - iS)^{-1}(I + iS)$ along $(-H(g^t(q), g^t(p), t, g^t(q))$.

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