Cohomology in Physics

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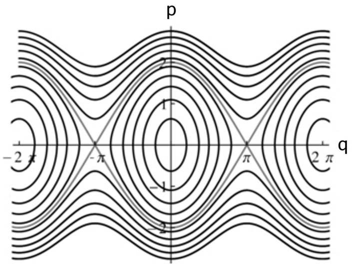
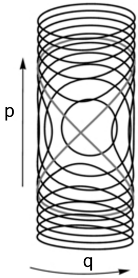
Symplectic Geometry

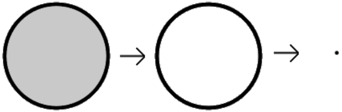




Phase Space (q,p) as a symplectic manifold with 2-form ω
Vector fields correspond to 1-forms $i_v\omega = \alpha_v$
Function $\rightarrow$ Fields/Flow

$$\{A, B\} \xrightarrow{i_v\omega = dA} [v_A, v_B]$$

Homology/Cohomology Intuition



$$\partial A = 0, \text{ Cycle}$$
$$A = \partial B, \text{ Boundary}$$
$$H_k = \text{Holes}$$

$$C \xrightarrow{grad} V \xrightarrow{curl} V \xrightarrow{div} C$$

$$dA = 0, \text{ Cocycle}$$
$$A = dB, \text{ Coboundary}$$
$$H^k = \text{?}$$

Vector Field Cohomology

$$\{A, B\} \xrightarrow{i_v\omega = dA} [v_A, v_B]$$

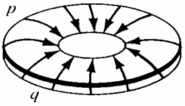
Noether:

$$v_A^\alpha \partial_\alpha H = 0 \iff v_H^\alpha \partial_\alpha A = 0$$
$$\frac{dA}{dt} = \{A, H\}$$

$i_v\omega = dH$, v is Hamiltonian

$di_v\omega = 0$, v is Symplectic

$H^1_{dR} = \frac{Sym}{Ham}$ = Fields not described by functions



Lie Algebra Cohomology

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \hookrightarrow & e & \twoheadrightarrow & [A, B] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R} & \hookrightarrow & \{A, B\} & \twoheadrightarrow & [v_A, v_B] & \longrightarrow & 0
\end{array}
$$

$$d\alpha(X) = \alpha([X, \cdot])$$
$$d\alpha(X, Y) = 0, \text{ Cocycle}$$
$$\alpha(X, \cdot) = d\beta(X), \text{ Coboundary}$$

Central Extension e of g by R via 2-cocycle α

$$[A + aC, B + bC] = [A, B] + \alpha(A, B)C$$

$H^2(g) =$ Nontrivial, Hamiltonian/Poisson

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